Fast and Slow solutions in General Relativity: The Initialization Procedure

Mirta S. Iriondo*, Enzo O. Leguizamón †
and Oscar A. Reula ‡
FaMAF, Medina Allende y Haya de la Torre,
Ciudad Universitaria, 5000 Córdoba, Argentina

February 7, 2008

Abstract

We apply recent results in the theory of PDE, specifically in problems with two different time scales, on Einstein's equations near their Newtonian limit. The results imply a justification to Postnewtonian approximations when initialization procedures to different orders are made on the initial data. We determine up to what order initialization is needed in order to detect the contribution to the quadrupole moment due to the slow motion of a massive body as distinct from initial data contributions to fast solutions and prove that such initialization is compatible with the constraint equations. Using the results mentioned the first Postnewtonian equations and their solutions in terms of Green functions are presented in order to indicate how to proceed in calculations with this approach.

1 Introduction

In recent papers there has been given a rigorous justification of the Newtonian limit approximation in General Relativity. In one case, [1], assuming symmetric hyperbolic equations for the matter (including appropriate boundary conditions for it), and in the other, [2], assuming Vlasov type matter, there has been shown that given a Newtonian solution, there exists a nearby general relativistic solution for a time intervall which is independent on the limiting parameter. The proofs of these results relay on ideas pioneered by H-O Kreiss on dynamical systems with

*Supported by STINT, The Swedish Foundation for International Cooperation in Research and Higher Education.
†Fellow of Se.CyT-UNC.
‡Member of CONICET.

different time scales which in turn relay on energy estimates for symmetric hyperbolic systems.

The above mentioned results are based on a initialization procedure, that is, the proximity of full relativistic solutions to the Newtonian ones is obtained by choosing the initial data in a very special way, basically ensuring that the time derivatives of the solutions at the initial surface stay bounded on that limit. This initialization procedure is only needed to a finite order on the limiting parameter (which is taken to be one over the speed of light) implying that fast behaving parts of solutions stay under control order by order along evolution.

In this paper we advance further into this problem by showing two things:

First that the General Relativity equations can be cast into a form on which the standard theory of different time scales applies (c.f. [3], [4], and [5]). This substantially improves the results in [1], for there there were singular terms outside the principal part which had to be dealt with in a very involved way. This gives further information on the behavior of the fast part of the solutions, that is the part that comes from the failure to initialize data to all orders. In particular, it gives the equations that the leading order fast behaving parts of the solutions satisfies and so its bulk behavior. This result should be important in studying the scattering of gravitational waves on slow varying sources.

To be able to apply these standard results (estimates) we shall assume that sources for Einstein's equations also satisfy symmetric hyperbolic equations which are regular in the limit and, if boundary conditions are needed to deal with them, that they are of such nature as to allow for the estimates to hold. Admittedly there are some problems to grant this to hold for some specific cases, but that is a problem on our understanding of the description of normal matter, and are not very much related to the dynamics of the gravitional degrees of freedom, so we do not address this issue further.

Second that the initialization procedure can be done to higher enough orders so that the quadrupole formula should follow. The initialization procedure implies some relations between initial data besides the one implied by the constraint equations, and so the above mentioned results implies that showing that certain elliptic systems of equations have solutions with the appropriate asymptotic behavior. The result is only an argument, for the method used does not allow us to have estimates, and so to control evolution, all the way up to some portion of future null infinity, the region where the quadrupole formula should hold. A rigorous result for that issue should follow by studying evolution either along null cones, as considered by Winicour, [6], or along asymptotically null surfaces as studied by Friedrich, [7].

The plan of the paper is as follows:

In the second Section we introduce the general theory of different time scales systems and quote the relevant Theorem. We then follow [1] and cast Einstein's equations as a symmetric-hyperbolic-elliptic system. By assigning units to the fields appearing in those equations one can determine how the parameter $\varepsilon = \frac{1}{c}$, where c is the velocity of light, appears in the system and in this way the solutions become

a one parameter family of solutions to Einstein's equations. A further rescaling of the fields is needed so that the structure of the equations is in the form assumed in the hypotesis of the mentioned Theorem.

In Section 3 we calculate the flux of energy coming from the slow part of the solution (i.e the contribution to the flux of energy due to the slow motion of a massive body) and the flux of energy coming from the initial data without sources, that is the contribution from fast solutions. We conclude that one should initialize up to order three in ε in order to isolate the contribution to the energy flux coming from the slow part of the solution. We further prove a Lemma stating that such initialization is consistent with the constraint equations and so that solutions with these characteristics exists.

In an Appendix we look at the solution to the initialized data to order ε and obtain an explicit solution in terms of the shift vector N^a and the source S^{ab}. This is done in order to illustrate how the general setting of the theory of different time scales produces the correct Postnewtonian equations to that order and how to proceed if one whishes to compute higher order corrections.

2 Preliminaries

The study of systems with different time scales reduces to the study of systems of partial differential equations which are singular in the limit of one of this time scales going to zero. The study of such a limit distinguishes between different classes of solutions according to their limiting behavior. Solutions which behave smoothly on the limit are called slow, they move according to the time scale which remains finite, those which do not have a well defined limit are called fast, they have dependence on the time scale we are setting to zero. As an example we consider the following system:

$$u_t = \frac{1}{\varepsilon}(u_x - v_x), \tag{2.1}$$

$$v_t = v_x. \tag{2.2}$$

Since the system is linear we can consider its Fourier modes, with the ansatz

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \mathrm{e}^{i(kx-\omega t)},$$

we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}(x,t) = u_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{ik(x+\frac{t}{\varepsilon})} + v_0 \begin{pmatrix} \frac{\varepsilon}{1-\varepsilon} \\ 1 \end{pmatrix} \mathrm{e}^{ik(x+t)}.$$

It is clear that the first term in this expression is a fast solution, while the second term is a slow one. Notice that for a solution to be slow it is necessary that its time derivative stay bounded, in particular at $t = 0$. One of the main results of the theory of different time scales is that, under some circumstances, this

condition is also sufficient. In particular this allow us to pick slow solutions from initial data requiring that a number of consecutive time derivatives at the initial time are bounded, this procedure is called initialization. Another interesting output of the theory is that it gives precise information on the evolution of the fast and slow parts.

We turn now to the general setting and quote the main results of the theory in the following Theorem.

Theorem 2.1 ([8], see also [9]) *Let*

$$A^0{}_{ij}(\varepsilon u^k)\frac{\partial}{\partial t}u^j = (\frac{1}{\varepsilon}K^a{}_{ij} + A^a_{1ij}(u^k,\varepsilon))\partial_a u^j + B_i(u^k,\varepsilon), \tag{2.3}$$

be a symmetric-hyperbolic system containing the small parameter ε; i.e., let the matrices K^a, A^a_1 and A^0 be symmetric and A^0 positive definite. Assume that the matrices A^a_1 and the vector B are continuous in ε uniformly in u for bounded u, and are C^s in u uniformly in ε with $s \geq s_0 \equiv [n/2]+2$ and that the initial data $u(0,x,\epsilon) = (q_{ab}(x,0), p_{ab}(x,0), r^a{}_{bc}(x,0))$ lies in $H^s(\mathbf{R}^n)$. Then the solution $u(t,x,\epsilon)$ exists for a time T independent of ε. Moreover if the initial data for (2.3) is

$$u(0,x,\epsilon) = u_0(x) + \varepsilon\, u_1(x) + \mathcal{O}(\varepsilon^2)$$

where

$$K^a\partial_a u_0 = 0,$$

then

$$u(t,x,\epsilon) = u^0(t,x) + \epsilon[v^0(t,x) + \tilde{u}(t,x,\epsilon)] + \mathcal{O}(\epsilon^2),$$

with u^0 and $\tilde{u}$ satisfying

$$\begin{aligned} A^0(0)u^0_t + A^a_1(u^0,0)\partial_a u^0 + K^a\partial_a v^0 &= B_0 \\ K^a\partial_a u^0 &= 0 \\ u^0(0,x) &= u_0(x) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} [A^0(0) + \varepsilon u^0 A^0_u(0)]\tilde{u}_t + A^a_1(u^0,0)\partial_a\tilde{u} + \frac{1}{\varepsilon}K^a\partial_a\tilde{u} &= B_1 \\ \tilde{u}(0,x) &= u_1(x) - v^0(0,x) \end{aligned} \tag{2.5}$$

respectively

$$\begin{aligned} B_0 &= B(t,x,u^0,0) \\ B_1 &= B_\varepsilon(t,x,u^0,0) + (v^0+\tilde{u})B_u(t,x,u^0) \\ &- (v^0+\tilde{u})A^a_u(u^0,0)\partial_a u^0 - A^0(0)v^0_t - A^a(u^0,0)\partial_a v^0, \end{aligned}$$

and v^0 chosen in such a way that $K^a\partial_a u^0 = 0$.

The solutions of (2.5) are fast solutions depending on the choice of the initial data. There exists further local (in $\mathbf{T}^n$) refinements of these results, that give us information about the behaviour of these fast solutions, see for example [10], [11] and [5]. These results should be an useful tool for the understanding of problems like the scattering of gravitational waves by slow moving bodies.

If we would now want that the fast solution appears in the second order of ε, u^0 and u^1 would have to satisfy equations similar to (2.4) and $\tilde{u}$ one similar to (2.5) (with different sources), so that u_t and u_{tt} remain bounded when $\varepsilon \to 0$ at $t = 0$. Summarizing we can suppress the fast part up to $\mathrm{O}(\varepsilon^n)$, demanding that n-time derivatives stay bounded when $\varepsilon \to 0$.This procedure is called *initialization*, for more details see [8] and [3]. Note that in the case of General Relativity the choice of initial data for the initialization procedure is not trivial for it must be consistent with the constraint equations.

In order to give a formulation of the Postnewtonian limit in General Relativity as an initialization procedure, we use the variables given in [1], such that the Einstein equations can be put as a symmetric hyperbolic-elliptic system. Defining the lapse function as

$$N \equiv \frac{1}{1-\varepsilon^2 U}, \tag{2.6}$$

with U being the Newtonian potential and

$$r^{ab}{}_c \equiv \frac{1}{2\varepsilon^3}(\partial_c q^{ab} - \frac{1}{2} q^{ab} q_{ed} \partial_c q^{ed}) \equiv \frac{1}{2\varepsilon^3 \sqrt{q}} \partial_c(\sqrt{q} q^{ab}) \tag{2.7}$$

$$p^{ab} \equiv \frac{1}{\varepsilon^2} \bar{\pi}^{ab}, \tag{2.8}$$

where ∂_c is the derivative operator associated to a flat three-metric e^{ab} and q_{ab} is the three metric induced on the hypersurfaces $t = const.$. The evolution equations become

$$\begin{aligned} \dot{q}^{ab} &= -8\varepsilon^2 q^{ab} N \dot{U} + 2\varepsilon^2(1-4\varepsilon^2 U) N^3 q^{\frac{1}{2}} (q^{ab} p - p^{ab}) \\ &\quad -2\varepsilon^2 q^{\frac{1}{2}} N \bar{D}^{(a} N^{b)} + 2\varepsilon^2 q^{ab} \bar{D}_c N^c, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \dot{p}^{ab} &= -q^{\frac{3}{4}} N^{\frac{1}{2}} \frac{1}{\varepsilon}(q^{cd} \partial_d r^{ab}{}_c - 2 q^{c(a} \partial_c r^{b)d}{}_d) - 2 q^{\frac{1}{2}} \frac{1}{\varepsilon^2} q^{ab}(\Delta U - \rho) \\ &\quad +2S^{ab} + \varepsilon^2 N^c \partial_c p^{ab} + \varepsilon F^{ab}(\varepsilon, r^{de}{}_c, p^{de}, \partial_c U, \partial_c N^d) \end{aligned} \tag{2.10}$$

$$\begin{aligned} \dot{r}^{ab}{}_c &= -\frac{N^{\frac{1}{2}}}{q^{\frac{1}{4}}} \frac{1}{\varepsilon}(\partial_c p^{ab} - 2\delta_c^{(a} \partial_d p^{b)d}) \\ &\quad +\frac{1}{\varepsilon}\{q^{ab} \partial_c \partial_d N^d - q^{d(b} \partial_c \partial_d N^{a)}\} \end{aligned}$$

$$+\frac{4N^{\frac{1}{2}}}{q^{\frac{1}{4}}}\frac{1}{\varepsilon}\delta_c^{(a}J^{b)}+\frac{2}{\varepsilon}q^{ab}\partial_c(N\dot{U})+\varepsilon^2N^d\partial_d r^{ab}{}_c$$
$$-2\varepsilon Nq^{ab}N^d\partial_c\partial_d U$$
$$+\varepsilon F^{ab}{}_c(\varepsilon, r^{ab}{}_c, p^{ab}, \partial_c U, \dot{U}, \partial_c N^a), \tag{2.11}$$

where N^a is the shift, $\bar{D}_a$ is the covariant derivative associated with $\bar{q}_{ab}$, $\Delta \equiv \bar{q}^{ab}\partial_a\partial_b$, $F^{ab} = F^{ab}(\varepsilon, r^{ab}{}_c, p^{ab}, \partial_c U, \partial_c N^a)$,
$F^{ab}{}_c = F^{ab}{}_c(\varepsilon, r^{ab}{}_c, p^{ab}, \partial_c U, \dot{U}, \partial_c N^a)$, and all F's in the equations are smooth pointwise functions of all their arguments and the quadratic terms in $r^{ab}{}_c$ and p^{ab} are the highest powers the factors will appear in and are $\mathcal{O}(\varepsilon^2)$.

In these variables, the constraint equations become

$$\Delta U - \frac{1}{2}\varepsilon\partial_c r^{cd}{}_d = \rho + \varepsilon^2 F(\varepsilon, r^{ab}{}_c, p^{ab}, \partial_c U) \tag{2.12}$$

$$-2\partial_c p^{ca} = 4J^a + \varepsilon^2 F^a(\varepsilon, r^{ab}{}_c, p^{ab}, \partial_c U) \tag{2.13}$$

$$r^{ab}{}_c = \frac{1}{2\varepsilon^3\sqrt{q}}\partial_c(\sqrt{q}q^{ab}). \tag{2.14}$$

The system (2.10)-(2.11) is symmetric hyperbolic, and has an energy estimate finite for fixed ε . In order to have an energy estimate bounded in the limit when $\varepsilon \to 0$ the authors in [1], choose a gauge (i.e. a selection of lapse and shift) that makes $r^{ab}{}_b \equiv 0$ and $\dot{r}^{ab}{}_b = 0$. As a matter of fact one could relax this gauge choice to $r^{ab}{}_b \equiv \mathcal{O}(\varepsilon^\beta)$ and $\dot{r}^{ab}{}_b = \mathcal{O}(\varepsilon^\beta)$ where β is a positive and arbitrary number that guarantees that the energy estimates stay bounded even when $\varepsilon \to 0$. In this paper we choose $\beta = 3$, the reason of this choice depends on the initialization procedure and it shall become clear in the next Section.

This choice implies the following equations for N^a:

$$\partial_c D^c N^b - \partial^b D_c N^c - 4(J^b + \partial^b N\dot{U}) + 2\varepsilon^2 q^{cb}N^d\partial_c\partial_d U - \varepsilon^2 G^b = 0 \tag{2.15}$$

$$D_d N^d = -2N\dot{U}, \tag{2.16}$$

combining equations (2.15) and (2.16) we get

$$\partial_c D^c N^b + \partial^b D_c N^c - 4J^b + 2\varepsilon^2 q^{cb}N^d\partial_c\partial_d U - \varepsilon^2 G^b = 0 \tag{2.17}$$

and the constraint equation (2.12) becomes

$$\Delta U = \rho + \varepsilon^2 F(\varepsilon, r^{ab}{}_c, p^{ab}, \partial_c U). \tag{2.18}$$

Thus the system given by (2.10), (2.11), (2.13), (2.14), (2.17) and (2.18) constitutes a symmetric hyperbolic-elliptic system, the hyperbolic and elliptic part are given by the equations (2.10), (2.11) and (2.17), (2.18) respectively.

By inspection one can see that the source $B_i(u^k,\varepsilon)$ in (2.10) and (2.11) takes the form $B_i(u^k,\varepsilon)=\frac{C_{ij}}{\varepsilon}u^j+F_i(u^k,\varepsilon)$, where C_{ij} is a constant matrix. Since we want to apply Theorem 2.1 on this system, the source has to be smooth in ϵ, thus we rescale the variables[1] in the following way.

$$\bar{r}_c^{ab}=\varepsilon r_c^{ab} \qquad \bar{p}^{ab}=\varepsilon p^{ab} \text{ and } q^{ab}=e^{ab}+\varepsilon^2\hat{h}^{ab}. \tag{2.19}$$

Since F^{ab} and F_c^{ab} have quadratic terms in the variables r_c^{ab} and p^{ab} , we obtain a system such as (2.3) with a source term depending smoothly on ε, namely:

$$\begin{aligned} \dot{q}^{ab} &= -8\varepsilon^2 q^{ab}N\dot{U}+2\varepsilon(1-4\varepsilon^2U)N^3q^{\frac{1}{2}}(q^{ab}\bar{p}-\bar{p}^{ab}) \\ &\quad -2\varepsilon^2q^{\frac{1}{2}}N\bar{D}^{(a}N^{b)}+2\varepsilon^2q^{ab}\bar{D}_cN^c, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \dot{\bar{p}}^{ab} &= -q^{\frac{3}{4}}N^{\frac{1}{2}}\frac{1}{\varepsilon}(q^{cd}\partial_d\bar{r}^{ab}{}_c-2q^{c(a}\partial_c\bar{r}^{b)d}{}_d)-2q^{\frac{1}{2}}\frac{1}{\varepsilon}q^{ab}(\Delta U-\rho) \\ &\quad +2\varepsilon S^{ab}+\varepsilon^2N^c\partial_c\bar{p}^{ab}+\varepsilon F^{ab}(\varepsilon,\bar{r}^{de}{}_c,\bar{p}^{de},\partial_cU,\partial_cN^d) \end{aligned} \tag{2.21}$$

$$\begin{aligned} \dot{\bar{r}}^{ab}{}_c &= -\frac{N^{\frac{1}{2}}}{q^{\frac{1}{4}}}\frac{1}{\varepsilon}(\partial_c\bar{p}^{ab}-2\delta_c^{(a}\partial_d\bar{p}^{b)d})+ \\ &\quad +q^{ab}\partial_c\partial_dN^d-q^{d(b}\partial_c\partial_dN^{a)} \\ &\quad +\frac{4N^{\frac{1}{2}}}{q^{\frac{1}{4}}}\delta_c^{(a}J^{b)}+2q^{ab}\partial_c(N\dot{U})+\varepsilon^2N^d\partial_d\bar{r}^{ab}{}_c \\ &\quad -2\varepsilon^2Nq^{ab}N^d\partial_c\partial_dU \\ &\quad +\varepsilon^2F^{ab}{}_c(\varepsilon,\bar{r}^{ab}{}_c,\bar{p}^{ab},\partial_cU,\dot{U},\partial_cN^a). \end{aligned} \tag{2.22}$$

In the same gauge as before, the constraint equations and the equation for the shift vector become

$$\Delta U=\rho+\varepsilon^2F(\varepsilon,\bar{r}^{ab}{}_c,\bar{p}^{ab},\partial_cU) \tag{2.23}$$

$$-2\partial_c\bar{p}^{ca}=4\epsilon J^a+\varepsilon^2F^a(\varepsilon,\bar{r}^{ab}{}_c,\bar{p}^{ab},\partial_cU) \tag{2.24}$$

$$\bar{r}^{ab}{}_c=\frac{1}{2\varepsilon^2\sqrt{q}}\partial_c(\sqrt{q}q^{ab}), \tag{2.25}$$

and

$$\partial_cD^cN^b+\partial^bD_cN^c-4J^b+3\varepsilon F^b+\varepsilon^2(4Nq^{cb}N^d\partial_c\partial_dU-F^{ab}{}_b)=0. \tag{2.26}$$

[1]A similar rescaling is used in [2].

From now on, to simplify the notation, we will work without bars on the variables p^{ab} and $r^{ab}{}_c$. As stated in the Introduction we assume that the above system is part of a bigger symmetric hyperbolic system which includes the equations for the sources, they are assumed to be regular in ε and of such a type that global estimates can be obtained.

The existence of smooth one parameter family of solutions is guaranteed because they prove that energy estimates corresponding to the hyperbolic part of the system stay bounded when $\varepsilon \to 0$ via a particular gauge choice, the elliptic variables satisfy a Gårding estimate in terms of the hyperbolic ones and the boundedness in time of the solutions are guaranteed via the initialization procedure.

Another interesting problem which can be tackled with these techniques is whether for each given fast Postnewtonian solution (i.e. a fixed slow background solution and a highly oscillating part), there exists a solution to the full Einstein equations in the *same intervall of existence* and that it remains close for that whole intervall to the first one in some norm. In other words, whether the solution to (2.3) exists and converges to the solution of the limit equation for as long as the General relativistic solution exists.

3 The Postnewtonian limit as an initialization procedure.

The Theorem quoted above imply that solutions to Einstein's equations, (the system given by (2.20) to (2.22)), can be splitted into slow and fast ones, and this is ruled by the choice of initial data.

In the context of General Relativity the slow part of the solution contains information about the matter fields that are moving with velocity much lower than c, that is with its own time scale, while the fast part contains information essentially due to the arbitrariness of initial data and so is not related directly to the sources.

In the context of symmetric-hyperbolic systems this means that on the Postnewtonian approximations the highly oscillatory part (fast part) of the solution should be suppressed up to higher order of ε. For example, if we initialize (choose initial data such that the time derivative of the solution is bounded independently of ε at $t = 0$) up to order ε the slow part appears at zeroth order and the fast one at first order. The degree of initialization needed depends on the problem at hand. In this Section we show that to get the quadrupole formula one necessary to initialize up to order ε^3. This guarantees that no contribution from fast solutions would be present in the gravitational radiation flux to that order. We then show that such initialization is possible.

To know at which order of ε we have to initialize, we estimate the contribution to the energy flux at $\mathcal{I}^+$ from both, sources and vacuum (pure initial data) solutions. Since we are only interested in the order of ε at which these contributions appear, it is enough to estimate them in linearized gravity, for example see [12].

To this end we keep the physical units in the Einstein equations (i.e $k = G \neq 1$), and we check the power at which ε appears in the radiation due to the matter (quadrupole contribution, with zero initial data) and in the radiation due to the initial data (without sources).

In this approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu},$$

and in the absence of matter the energy flux in the x^1 direction due to the initial data can be calculated as

$$ct^{01} = \frac{1}{32\pi k\varepsilon^3}[\delta\dot{g}_{23}^2 + \frac{1}{4}(\delta\dot{g}_{22} - \delta\dot{g}_{33})^2],$$

where the dot means the time derivative. In the presence of matter, with initial data $\delta g_{\mu\nu} = \delta\dot{g}_{\mu\nu} = 0$, we can use the same formula as above because at large distance from the bodies the matter waves can be considered as plane waves. By (2.19) $\delta g_{ij} = \varepsilon^2\hat{h}_{ij}$, the flux of energy becomes

$$ct^{01} = \frac{\varepsilon}{32\pi k}[\dot{\hat{h}}_{23}{}^2 + \frac{1}{4}(\dot{\hat{h}}_{22} - \dot{\hat{h}}_{33})^2],$$

Thus choosing the initial data $\hat{h}_{ij} = \varepsilon^\alpha f_{ij}$, the flux becomes $ct^{01} = \mathcal{O}(\varepsilon^{2\alpha+1})$, on the other side the flux due to the matter is calculated as $ct^{01} = \mathcal{O}(\varepsilon^5)$ (see the Appendix). We conclude that initializing up to order ε^3 the fast part of the solution will contribute at a higher order of ε to the flux. It means that the initial data will be choosen as

$$u|_{t=0} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \mathcal{O}(\varepsilon^4).$$

This is equivalent to demand that the first four time derivatives of the solution stay bounded in $t = 0$ when $\varepsilon \to 0$ (c.f. [3]).

Besides an hyperbolic system Eintein's equations also include a constraint system, that is, equations which relate the otherwise arbitrary initial data fields between each other. Thus it must be chequed that these constraints are consistent with the initialization procedure. To do that we add to the equations arising from the initialization the constraint equations and show that in the resulting system, the hierarchy of elliptic systems admits solutions with the correct asymptotic fall off.

Defining the pseudotensor h^{ab}

$$\sqrt{q}q^{ab} = \sqrt{e}e^{ab} + \varepsilon^2 h^{ab} \tag{3.1}$$

and from the definition of $r^{ab}{}_c$ given by (2.25), we obtain

$$r^{ab}{}_c = \frac{1}{2\sqrt{q}}\partial_c(h^{ab}).$$

Choosing the gauge condition $r^{ab}{}_b = 0$ and $\dot{r}^{ab}{}_b = 0$ up to order ε^3 and demanding the boundedness of the first four time derivatives we get that $p_i{}^{ab}$ and $h_i{}^{ab}$, $i = 0, 1, 2, 3$ should satisfy at $t = 0$, the following differential equations:

$$
(3.2)\qquad
\begin{array}{rcl}
\Delta_0 {h_0}^{ab} & = & 0,\\
\partial_c {p_0}^{ab} & = & 0,\\
\Delta_0 {h_1}^{ab} & = & 0,\\
\Delta_0({p_1}^{ab} + \partial^{(a}{N_0}^{b)}) & = & 0,\\
\Delta_0({p_2}^{ab} + \partial^{(a}{N_1}^{b)}) & = & 0,\\
\Delta_0 {h_2}^{ab} & = & 2{S_0}^{ab} + {F_0}^{ab} + \partial^{(a}{\dot{N}_0}{}^{b)} - e^{ab}F_0,\\
\Delta_0({p_3}^{ab} + \partial^{(a}{N_2}^{b)}) & = & -2{\dot{S}_0}{}^{ab} - {\dot{F}_0}{}^{ab} + 2e^{ab}\dot{F}_0 - \partial^{(a}{\ddot{N}_0}{}^{b)} + \\
 & & +\partial^d {F_0}^{ab}{}_d - \partial^{(a}{F_1}^{b)} - 2e^{ab}\partial^e({N_0}^d\partial_e\partial_d U_0),\\
\Delta_0 {h_3}^{ab} & = & 2{S_1}^{ab} - {F_1}^{ab} + \partial^{(a}{\dot{N}_1}{}^{b)} - 2e^{ab}F_1,
\end{array}
$$

where $\Delta_0 = e^{ab}\partial_a\partial_b$. These equations are consistent with the constraint up to order ε^3, for they have been included in the set.

Lemma 3.1 *The equation system for the initial data given by (3.2) admits a solution.*

Proof: From the first five equations, we can choose

$$
(3.3)\qquad {p_0}^{ab} = 0,\ {r_0}^{ab}{}_c = 0,\ {r_1}^{ab}{}_c = 0,\ {p_1}^{ab} = -\partial^{(a}{N_0}^{b)}
$$

and

$$
(3.4)\qquad {p_2}^{ab} = -\partial^{(a}{N_1}^{b)}.
$$

Therefore, in order to guarantee the existence of solutions, it is sufficient to prove that the sources in the three last equations in (3.2) have the decay $\mathcal{O}(\frac{1}{r^3})$ (c.f.[13]).

Due to the choice made in (3.3), the functions ${F_0}^{ab}$, F_0, ${F_0}^{ab}{}_c$, F_1, ${F_1}^a$ and ${F_1}^{ab}$ are all linear functions of $\partial_a(U_0{N_0}^b)$. Similarly $\dot{F}_0$ and ${\dot{F}_0}{}^{ab}$ depends linearly on $\partial_t\partial_a(U_0{N_0}^b)$. But the vector fields ${N_0}^a$ and ${N_1}^a$, and the function U_0 satisfy the following differential equations:

$$
\begin{array}{rcl}
\Delta_0{N_0}^a + \partial^a\partial_c{N_0}^c & = & 4{J_0}^a,\\
\Delta_0{N_1}^a + \partial^a\partial_c{N_1}^c & = & 4{J_1}^a,\\
\Delta_0 U_0 & = & \rho_0.
\end{array}
$$

Thus, because of the compactness of the support of the sources, the standard theory of elliptic systems in $\mathbf{R}^n$ (c.f. [13]) assures that U_0, ${N_0}^a$ and ${N_1}^a$, and their time and spatial derivatives decay as $\mathcal{O}(\frac{1}{r})$ and $\mathcal{O}(\frac{1}{r^2})$ respectively. Hence we conclude that the sources decay fast enough as to ensure existence of solutions for ${h_2}^{ab}$, ${h_3}^{ab}$, and ${p_3}^{ab} + \partial^{(a}{N_2}^{b)}$.

In order to have a well defined ${p_3}^{ab}$, it remains to prove that ${N_2}^a$ exists. Note that the gauge condition implies that ${N_2}^a$ must satisfy

$$\Delta_0 N_2{}^a + \partial^a \partial_c N_2{}^c = 4J_2{}^a - 3F_1{}^a - 4N_0{}^d \partial_d \partial^a U_0 + 2F_0{}^{ab}{}_b,$$

and that the source in this equation has a fast enough decay. Thereby the existence of $N_2{}^a$ and so of $p_3{}^{ab}$ is assured.□

This Lemma proves that we can initialize at least up to order ε^3, hence by Theorem 2.1 we can assure that the fast solution appears in a higher order than ε^3

Appendix: The evolution of the initialized data

In order to exemplify the initialization procedure, we initialize up to order ε and solve explicitly the equation for u^0, u^1 and v^0, choosing the initial data u_0 and u_1 as in (3.3). In fact this choice gives that $\tilde{u}$ becomes a slow solution suppressing the fast one to a higher order of ϵ.

Claim: *Given the initial data*

$$\begin{aligned} u(0,x) &= \varepsilon \begin{pmatrix} p_1^{ab} \\ r_{1c}{}^{ab} \end{pmatrix} (x) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon \begin{pmatrix} -\partial^{(a} N^{b)} \\ 0 \end{pmatrix} (x) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

where $N^a(x)$ is the solution of the elliptic equation (3.6) for $t = 0$, the solution of (2.21) and (2.22) can be written as

$$\begin{aligned} u(t,x) &= \varepsilon \begin{pmatrix} p_1^{ab} \\ r_{1c}{}^{ab} \end{pmatrix} + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon \begin{pmatrix} \tilde{p}^{ab} + v^{ab} \\ \tilde{r}^{ab}{}_c + v^{ab}{}_c \end{pmatrix} + \mathcal{O}(\varepsilon^2), \end{aligned}$$

with

$$v = \begin{pmatrix} -\partial^{(a} N^{b)} \\ 0 \end{pmatrix}$$

and

$$\begin{aligned} \tilde{p}^{ab} &= \int_{|\mathbf{x}-\mathbf{x}'| \leq \frac{t}{\varepsilon}} \frac{\Gamma_1^{ab}(t - \varepsilon|\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x' + \partial^{(a} N^{b)} \\ \tilde{r}_c^{ab} &= \varepsilon \partial_c \int_{|\mathbf{x}-\mathbf{x}'| \leq \frac{t}{\varepsilon}} \frac{\Gamma_2^{ab}(t - \varepsilon|\mathbf{x}-\mathbf{x}'|, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \mathrm{d}^3x', \end{aligned}$$

where $\Gamma_1^{ab} = -(\Delta \partial^{(a} N^{b)} + 2\varepsilon^2 \dot{S}^{ab})$ and $\Gamma_2^{ab} = \partial^{(a} \dot{N}^{b)} + 2S^{ab}$, ρ and J^a have compact support in R^3. Furthermore, the asymptotic behavior of $\tilde{u}$ is $\mathcal{O}(\frac{1}{r})$.

Proof: Since $u_0 = 0$ and thereby $K^a \partial_a u_0 = 0$, Theorem 2.1 assures that

$$u(t,x,\epsilon) = u^0(t,x) + \epsilon[v(t,x) + \tilde{u}(t,x,\epsilon)] + \mathcal{O}(\epsilon^2).$$

By applying (2.4) and (2.5) on the equations system given by (2.21) and (2.22), we obtain that u^0, and v will satisfy

$$\begin{aligned}
\dot{p}_0^{ab} + \partial^c v^{ab}{}_c - 2\partial^{(a} v^{b)d}{}_d &= 0 \\
\dot{r}_{0c}{}^{ab} + \partial_c v^{ab} - 2\delta_c^{(a} \partial_d v^{b)d} &= -\partial_c \partial^{(b} N^{a)} + 4\delta_c^{(a} J^{b)}, \\
K^a \partial_a u^0 &= 0.
\end{aligned} \tag{3.5}$$

At this order of ε the constraint equations and the gauge become

$$\begin{aligned}
\Delta U &= \rho \\
-2\partial_c p_0^{ca} &= 0 \\
r_{0c}{}^{ab} &= \frac{1}{2}\partial_c h_0^{ab} \\
\partial_c \partial^c N^b + \partial^b \partial_c N^c &= 4J^b,
\end{aligned} \tag{3.6}$$

where h^{ab} is a pseudotensor defined in (3.1). Choosing

$$v = \begin{pmatrix} -\partial^{(a} N^{b)} \\ 0 \end{pmatrix},$$

the system (3.5) becomes

$$\begin{aligned}
u_t^0 &= 0 \\
K^a \partial_a u^0 &= 0 \\
u_0(x) &= 0,
\end{aligned} \tag{3.7}$$

which is trivially satisfied by $u^0(t,x) = 0$.

Considering now the equation for $\tilde{u}$, namely

$$\tilde{u}_t + \frac{1}{\varepsilon} K^a \partial_a \tilde{u} = B_1 \tag{3.8}$$

$$\tilde{u}(0,x) = u_1(x) - v^0(0,x) = 0$$

with

$$B_1 = \begin{pmatrix} 2S^{ab} + \partial^{(a} \dot{N}^{b)} \\ 0 \end{pmatrix},$$

we obtain the following system

$$\dot{\tilde{p}}^{ab} + \frac{1}{\varepsilon}\partial^c \tilde{r}^{ab}{}_c = 2S^{ab} + \partial^{(a}\dot{N}^{b)} \tag{3.9}$$

$$\dot{\tilde{r}}^{ab}{}_c + \frac{1}{\varepsilon}\partial_c \tilde{p}^{ab} = 0 \tag{3.10}$$

and the constraint and the gauge become

$$\begin{aligned} \Delta U &= \rho \\ \partial_c p_1^{ca} &= -2J^a \\ r_{1c}{}^{ab} &= \frac{1}{2}\partial_c h_1^{ab} \\ \partial_c\partial^c N^b + \partial^b\partial_c N^c &= 4J^b. \end{aligned} \tag{3.11}$$

Since $\tilde{p}^{ab} = p_1^{ab} - v^{ab}$ and $\tilde{r}_c{}^{ab} = r_{1c}{}^{ab}$, instead of (3.9) we consider the differential equations

$$\begin{aligned} \dot{p}_1^{ab} + \frac{1}{\varepsilon}\partial^c r_{1c}{}^{ab} &= 2S^{ab} \\ \dot{r}_{1c}{}^{ab} + \frac{1}{\varepsilon}\partial_c p_1^{ab} &= -\frac{1}{\varepsilon}\partial_c\partial^{(a}\dot{N}^{b)}. \end{aligned} \tag{3.12}$$

The equations system (3.12) is clearly consistent with the gauge (3.11). Taking the time derivative to the system above, we obtain

$$\begin{aligned} \Box p_1^{ab} &= -(\Delta\partial^{(a}N^{b)} + 2\varepsilon^2\dot{S}^{ab}) \\ \Box h_1^{ab} &= 2\varepsilon(\partial^{(a}\dot{N}^{b)} + 2S^{ab}) \\ &:= 2\varepsilon\Gamma_2^{ab} \end{aligned} \tag{3.13}$$

and the solution of the second equation in (3.13) can be written as

$$h_1^{ab} = 2\varepsilon\int_{|\mathbf{x}-\mathbf{x}'|\leq\frac{t}{\varepsilon}} \frac{\Gamma_2^{ab}(t-\varepsilon|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3x'.$$

Because of the decay of the source, when $|\mathbf{x}| \gg |\mathbf{x}'|$, we write $|\mathbf{x}-\mathbf{x}'| \approx |\mathbf{x}| - \hat{n}\cdot\mathbf{x}'$, where $\hat{n} = \frac{\mathbf{x}}{|\mathbf{x}|}$. Hence in this first approximation we have

$$h_1^{ab} \approx \frac{2\varepsilon}{r}\int_{|\mathbf{x}-\mathbf{x}'|\leq\frac{t}{\varepsilon}} \Gamma_2^{ab}(t-\varepsilon r+\varepsilon\hat{n}\cdot\mathbf{x}',\mathbf{x}')\mathrm{d}^3x',$$

similarly we calculate p_1^{ab}.$\Box$

Following this procedure we can generate the solution that has as initial data (3.2) and the functions $u^2(t,x)$ and $u^3(t,x)$ obey similar equations than $u^0(t,x)$ and $u^1(t,x)$ do.

Remark 3.1 *In the calculus of the flux of energy due to the matter fields, we use the pseudo-tensor field calculated above instead of the spatial components of the perturbation of the metric. The difference between them can be calculated using equations (2.19) and (2.25), and because they have the same expansion in ε they can be considered as equal.*

References

[1] *On the Newtonian limit of General Relativity*, S. Fritelli and O. A. Reula, Comm. Math. Phys., **166**, 2, 1994.

[2] *The Newtonian Limit for Asymptotically Flat Solutions of the Vlasov-Einstein system*, Alan D. Rendall, Comm. Math. Phys., **163**, 1994, 89-112.

[3] *Problems with different time scales for nonlinear partial differential equations*, G. Browning and H. O. Kreiss, SIAM J. Appl. Math, **42**, 704-718, 1982.

[4] *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*,Majda A., Springer-Verlag, New York, 1984.

[5] *The Fast Singular Limit of Hyperbolic PDEs*, S. Schochet, J. Diff. Eq., **114**, 1994, 476-512.

[6] *Newtonian gravity on the null cone*, Winicour J., J. Math. Phys., **24**, 1193-1198, 1983.

[7] *Cauchy problems for the conformal vacuum field equations in General relativity*, Friedrich H., Comm. Math. Phys., **91**, 445-472, 1983.

[8] *Asymptotics for Symmetric Hyperbolic Systems with a large parameter*, S. Schochet, J. Diff. Eq., **75**, 1988, 1-26.

[9] *Compressible and incompressible fluids*, S. Klainerman and A. Majda, Comm. Pure Appl. Math, **35**, 629-653, 1982.

[10] *Remarques sur l'optique geométrique nonlinéaire multidimensionelle* , J.L. Joly, G. Métivier and J. Rauch, Séminar, Ecole Polytecnique, Palaiseau, 1990.

[11] *Coherent and Focusing multidimensional nonlinear geometric optics*, J.L. Joly, G. Métivier and J. Rauch, Centre de Recherche en Mathématiques de Bordeaux, preprint, june 1992.

[12] *The Classical Theory of Fields*, L. Landau and E. Lifshitz, edited by Addison-Wesley Publishing Company, Inc., 2nd edition, 1959.

[13] *On elliptic systems in* $\mathbf{R}^n$, R. Lockhart and R. McOwen, Acta Matematica, 1983, 125-135.